Once Again On the Klein Paradox

N.Kevlishvili[1], A. Khelashvili [1,2], T. Nadareishvili[1,2]

[1]*Department of Theoretical Physics,Tbilisi State University, Tbilisi, Georgia*
[2]*High Energy Physics Institute, Tbilisi, Georgia*

Abstract

After the short survey of the Klein Paradox in 3-dimensional relativistic equations, we present a detailed consideration of Dirac modified equation, which follows by one particle infinite overweighting in Salpeter Equation. It is shown, that the separation of angular variables and reduction to radial equation is possible by using standard methods in momentum space.
The kernel of the obtained radial equation differs from that of spinless Salpeter equation in bounded regular factor. That is why the equation has solutions of confined type for infinitely increasing potential.

1 Introduction

It is well known, that the Dirac equation in central symmetric field, which is a zero component of the Lorentz four-vector and is infinitely increasing, has a pathological property - the leakage through the infinite wall occurs or equivalently, the wave function has an oscillating (i.e. non normalizable) asymptotic behaviour. This unusual property is known as the Klein Paradox [1].The same drawback has two-particle Dirac (Breit) equation [2, 3]. In case of quarks and antiquarks this problem is usually avoided by introducing additional Lorentz-scalar interaction [4, 5]. Besides this, particular importance is given to the equal mixture of scalar and vector potentials, which excludes the spin-orbital coupling in Dirac equation [3, 5] and avoids the Klein Paradox in Breit equation as well, [2, 3, 6].

It seems quite natural to ask for relativistic covariant QFT equations for bound states such as Bethe-Salpeter (BS) equation. There is an opinion

that only vector exchange interaction is not desirable here due to the Klein paradox. But as far as we know this fact has not been proved regularly for BS equation yet. At the same time, it is clear that if the Klein Paradox exists here, the situation would be rather vague because the fundamental particles that transfer interaction between quarks are vectorial gluons and they must provide confinement of quarks into hadrons. Otherwise all attempts to obtain singular (f.e. q^{-4}) behaviour of gluon propagator [7] in the infrared area would be useless.

In recent years a lot of articles have been dedicated to 3-dimensional relativistic equations which follow from BS equation by using various reduction methods. These equations are interesting because they give the possibility to study quarkonium spectra for different potentials.

Despite the fact that 3-dimensional relativistic equations have a long history of study in recent years much attention has been paid to the formulations in which by infinite overwheighting of one of the constituent particles the problem is reduced to the Dirac equation for light particle in external field [10].

This paradigm is not clear to us, because in our opinion, the information on second quantization could not be lost completely whereas one of the particles becomes infinitely heavier. Really when we use 3-dimensional (for example, instantaneous) kernel in BS equation the latter is reduced to Salpeter equation which differs from two-particle Dirac (Breit) equation by projection operators in interaction term only on positive and negative frequencies, but do not contain their interference. These operators are the only remnants of secondary quantization and they cannot be neglected.

However in the 1960-s the difference caused by projection operator for mesons in quark model was found. For instance, P. Horwitz [11] in case of square wall potential has shown that if one keeps only v^2/c^2 terms in projection operators then some unpleasant features of Breit equation disappear, as a consequence of correct projecting, which excludes transitions from positive to negative frequency states. That is why the effective potential produced in this problem has correct sign and is free from unphysical singularities.

Although this result had been derived only in v^2/c^2 order, it was, by our knowledge, the first indication that there is a puzzle practically equivalent to the Klein paradox.

Later the whole attention was carried on Salpeter equation. It was supposed [3, 6] that QFT equations must be free from the Klein paradox. There are remarkable works [12, 13] dedicated to the study of Salpeter equation from this point of view. However final conclusions are made based on the above mentioned modified Dirac equation. This subject was considered also

in many other publications [14, 15, 16].

For example, the author of ref. [12] studied the following Salpeter equation

$$[E - (\vec{\alpha}^{(1)}\vec{p} + \beta^{(1)}m_1) - (-\vec{\alpha}^{(2)}\vec{p} + \beta^{(2)}m_2) - \Pi(\vec{p})V(r)]\psi(\vec{r}) = 0 , \qquad (1)$$

where $\Pi(\vec{p})$ is the well known projection operator

$$\Pi(\vec{p}) = \Lambda_+^{(1)}(\vec{p})\Lambda_+^{(2)}(-\vec{p}) - \Lambda_-^{(1)}(\vec{p})\Lambda_-^{(2)}(-\vec{p}) , \qquad (2)$$

and

$$\Lambda_\pm^{(i)}(\vec{p}) = \frac{\omega_p \pm (\vec{\alpha}^{(i)}\vec{p} + \beta^{(i)}m_i)}{2\omega_p}, \qquad \omega_p = \sqrt{\vec{p}^2 + m^2}$$

are corresponding projection operators for free spinors. They represent non-local integral operators in coordinate space.

In the limit when $m_1 = m_2 = m \to \infty$ the kernel of this integral representation (Bessel $K_0(m|\vec{r} - \vec{r'}|)$ function) transforms into local $\delta(\vec{r} - \vec{r'})$ function, then it is not difficult to determine the asymptotic behavior of wave function when $r \to \infty$. It falls exponentially for infinitely increasing potentials, V(r).

In the other work [13] the case $m_2 \to \infty$ is considered, when equation (1) is reduced to modified Dirac equation, describing a lighter particle's motion in "projected" potential:

$$[\vec{\alpha}\vec{p} + \beta m + \Lambda_+(\vec{p})V(r)]\psi(\vec{r}) = E\psi(\vec{r}) \qquad (3)$$

By means of the above mentioned integral representation and some additional asymptotic restrictions on the wave function and potential the author shows, that modified Dirac equation is free from the Klein paradox.

In our opinion all these statements are somewhat unsatisfactory, because they are based on some approximations and/or simplified assumptions. One thing is clear from these works - if modified Dirac equation is free from the Klein paradox, the same will be true for Salpeter equation.

Further we will consider our problem in momentum space and show that modified Dirac equation (3) has only discrete spectrum provided Schrodinger equation had such spectrum for the same potentials V(r). In our opinion, this statement is equivalent to the absence of the Klein paradox.

2   Modified Dirac Equation in Foldy-Wouthuyson Representation

Consequently, let us study modified Dirac equation (3). If we successively act on this equation with projecting operators $\Lambda_{\pm}(p)$, we get the following equation

$$[E - (\vec{\alpha}\vec{p} + \beta m)]\Lambda_+(\vec{p})\psi(\vec{r}) = \Lambda_+(\vec{p})V\psi(\vec{r}); \qquad (4)$$

and the additional constraint

$$\psi_- \equiv \Lambda_-\psi = 0 \qquad (5)$$

Hence equation (4) looks like

$$[E - (\vec{\alpha}\vec{p} + \beta m)]\Lambda_+\psi = \Lambda_+ V\Lambda_+\psi; \qquad (6)$$

Thus, the task is reduced to the problem on eigenfunctions and eigenvalues of the following hermitian Hamiltonian

$$H = (\vec{\alpha}\vec{p} + \beta m) + \Lambda_+ V\Lambda_+; \qquad (7)$$

It is convenient to use the Foldy-Wouthuysen (FW) transformation [17]

$$e^{iS}(\vec{\alpha}\vec{p} + \beta m))e^{-iS} = \beta\omega_p, \qquad \omega_p = \sqrt{\vec{p}^2 + m^2} \qquad (8)$$

It is clear, that

$$e^{iS}\Lambda_+(\vec{p}) = \sqrt{\frac{2\omega_p}{\omega_p + m}}\frac{1}{2}(1 + \beta)\Lambda_+(\vec{p})$$

$$\Lambda_+(\vec{p})e^{-iS} = \sqrt{\frac{2\omega_p}{\omega_p + m}}\Lambda_+(\vec{p})\frac{1}{2}(1 + \beta) \qquad (9)$$

For the transformed $\psi_{FW} = e^{iS}\psi$ wave function we get the following equation

$$(E - \beta\omega_p)\psi_{FW} = \qquad (10)$$
$$= \sqrt{\frac{2\omega_p}{\omega_p + m}}\frac{1}{2}(1 + \beta)\Lambda_+(\vec{p})\int d^3k V(\vec{p} - \vec{k})\Lambda_+(\vec{k})\frac{1}{2}(1 + \beta)\sqrt{\frac{2\omega_k}{\omega_k + m}}\psi_{FW}(k)$$

Now it is natural to use the 2-component representation

$$\psi_{FW} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{11}$$

Then equation (10) transforms to the following system:

$$(E - \omega_p)\varphi(\vec{p}) =$$
$$= \sqrt{\frac{2\omega_p}{\omega_p+m}}\frac{1}{2}(1+\beta)\Lambda_+(\vec{p})\int d^3k V(\vec{p}-\vec{k})\Lambda_+(\vec{k})\sqrt{\frac{2\omega_k}{\omega_k+m}}\varphi(\vec{p}); \tag{12}$$
$$(E + \omega_p)\chi(\vec{p}) = 0$$

As far as $E \neq -\omega_P$, the second equation has only trivial solution $\chi = 0$.

Let us calculate the matrix structure more explicitly on the right-hand side of equation (12). We get

$$(E - \omega_p)\varphi(\vec{p}) = \tag{13}$$
$$= \sqrt{\frac{2\omega_p}{\omega_p+m}}\int d^3k [\frac{\omega_p+m}{2\omega_p}V(\vec{p}-\vec{k})\frac{\omega_k+m}{2\omega_k} + \frac{\vec{\sigma}\vec{p}}{2\omega_p}V(\vec{p}-\vec{k})\frac{\vec{\sigma}\vec{k}}{2\omega_k}]\sqrt{\frac{2\omega_k}{\omega_k+m}}\varphi(\vec{p})$$

Let us make a remark that the same problem with the Hamiltonian (7) has been considered earlier by J.Sucher [14] . The author using transformations close to the above mentioned had obtain the equation similar to (13) except the difference in some kinematical factors. To our opinion after all author made somewhat premature conclusion: "Since these corrections do not dominate the effective interaction, one expects that there are normalizable solutions both in the scalar case and in the vector case".

Below we bring alternative (and practically equivalent) proof for the fourth component of the vector case. As it was mentioned in the introduction, just vector case is the principle one. In addition we remark that due to the $\frac{1+\beta}{2}$ factor scalar potential is actually presented in our study.

For this purpose we use the separation of angular variables and correspondingly, one dimensional radial equation in momentum representation, in which one additional nontrivial property of this equation will appear.

3 The radial form of modified Dirac equation

An angular analysis can be easily performed in standard way. For this purpose we should choose the basis of spherical spinors [18]

$$\varphi(\vec{p}) = f(p)\Omega_{jlM}(\vec{n_p}), \quad \vec{n_p} = \frac{\vec{p}}{p} \tag{14}$$

where Ω_{jlM} functions satisfy the following equations

$$(\vec{\sigma}\vec{n_p})\Omega_{jlM}(\vec{n_p}) = -\Omega_{jl'M}(\vec{n_p}), \quad l + l' = 2j \tag{15}$$

These functions are expressed in explicit form by spherical harmonics and corresponding Klebsh-Gordan coefficients:

$$\Omega_{jlM}(\vec{n}) = \begin{pmatrix} C^{jM}_{l,M-1/2,1/2,1/2} & Y_{l,M-1/2}(\vec{n}) \\ C^{jM}_{l,M+1/2,1/2,-1/2} & Y_{l,M+1/2}(\vec{n}) \end{pmatrix} \tag{16}$$

Then we present $V(\vec{p} - \vec{k})$ as series of spherical harmonics

$$V(\vec{p} - \vec{k}) = \sum_{l=0}^{\infty} \sum_{m'=-l}^{l} v_l(p,k) Y_{lm'}(\vec{n_p}) Y^*_{lm'}(\vec{n_k}) \tag{17}$$

and insert everything into the equation (13). Accounting orthogonality and completeness properties it is easy to verify that angular dependence is avoided and we get the following radial equation in momentum space

$$(E - \omega_p)f(p) = \int_0^{\infty} k^2 dk v_l(p,k)\chi(p,k)f(k) \tag{18}$$

where $\chi(p,k)$ is an additional factor resulting from projection operator and FW transformation,

$$\chi(p,k) = \sqrt{\frac{\omega_p + m}{2\omega_p}}[1 + \frac{pk}{(\omega_p + m)(\omega_k + m)}]\sqrt{\frac{\omega_k + m}{2\omega_k}} \tag{19}$$

Equation (18) is the main result of the present paper.

It seems that there is some unexpectedness in this equation. Particularly the fact that in the equation (13) matrix structure

$$\frac{\vec{\sigma}\vec{p}}{2\omega_p} V(\vec{p} - \vec{k}) \frac{\vec{\sigma}\vec{k}}{2\omega_k}$$

has been reduced completely and only the dependence on orbital-angular momentum remains. The non-total diagonalization was the more expectative outcome. But it seems that the projection operators $\Lambda_+(p)$ tear the spin-orbital coupling and restore the symmetry under usual 3-dimentional rotations.

The argument for this statement may be the observation derived in the recent years in Dirac equation with $V_v = \pm V_s$ potential (or $\frac{1\pm\beta}{2}$ projectors), the so called pseudospin symmetry [19, 20]. Anyway, the symmetries of the Hamiltonian under consideration need more detailed investigation.

4 Properties of spectrum of radial equation

The type of spectrum of radial (integral) equation (18) depends on the properties of its kernel, which now contains FW-factor $\chi(p, k)$ together with radial component of potential. Just this factor expresses nonlocality of effective potential.

First of all let us list several properties of $\chi(p, k)$ factor:

1) It represents the sum of two terms factorized in p and k variables. This property could be important during the study of various separable potentials.

2) When k=p (on the energy-shell in elastic scattering problem), $\chi = 1$.

3) It is positively-definite and bounded when $p, k \to \infty$.

4) It has no singularity for physical values of variables.

5) To reach the nonrelativistic limit in equation(13) it is enough to expand it into the powers of small p^2 and k^2. This operation does not imply smallness of potential V in contrast to the case of pure Dirac equation.

Based on these properties we can answer our main (principal) question if the Klein paradox takes place or not.

It is clear, that integral equation (18) without $\chi(p, k)$ is a radial form of Salpeter spinless equation in momentum space.

As it is known [21] Salpeter spinless equation for infinitely increasing potentials has confinement like solutions, i.e. only discrete spectrum. Due to the above mentioned properties of $\chi(p, k)$ the kernel of our equation is a product of completely continuous type kernel on bounded nonsingular functions. Therefore according to the well-known theorem [22, 23] the obtained kernel will be completely continuous type, as well. We can conclude that the spectrum of our equation (12) will be only discrete for such potentials if for those the Schrodinger (or spinless Salpeter) equation had the discrete spectrum (see, e.g. Theorem 7.8.3 in book [24]).

This statement, in our opinion, is equivalent to the absence of Klein paradox in modified Dirac equation, because it means that the primary equation has only normalizable solutions.

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